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FORCED OSCILLATIONS OF A NON-LINEAR SYSTEM WITH A REPULSIVE POSITIONAL FORCE*

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Non-linear systems with one degree of freedom, in which the positional force is directed away from the equilibrium position of the system, are considered. The existence of forced periodic oscillations, their Lyapunov stability, and the behaviour of amplitude-frequency characteristics are investigated. It is shown that stable periodic oscillations are possible in the case when the positional force has non-monotonic properties. Forced oscillations of a pendulum with respect to the upper equilibrium position are considered as an example.

Systems with repulsive positional forces appear not to have been previously considered in the literature. The well-known analytical methods of non-linear mechanics (/1, 2/ etc.) are based on the assumption of the nearness of the solutions under investigation to solution of the corresponding autonomous system, and are inapplicable to our systems because there are no periodic generating solutions. In this paper a qualitative investigation is made of

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the above-mentioned systems and the properties of the solutions under consideration are established directly from the form of the forcing term and the properties of the positional force.

1. Consider the system

$$x^{\cdot\cdot} + f(x) + \varepsilon p(\omega t) = 0 \tag{1.1}$$
$$p(\omega t) = -p(-\omega t) = p(\omega t + 2\pi)$$
$$f(x) = -f(-x),\ f(x)x < 0 \text{ for } x \neq 0 .$$

Because of the last condition the positional force is always directed so as to deflect the system, i.e. it is repulsive /3/. In the associated autonomous system $(\varepsilon = 0)$ periodic oscillations are impossible; $x = 0$ is an unstable position of equilibrium. We shall consider odd periodic oscillations about this position with period $2T = 2\pi/\omega$, equal to the period of the forcing term $(x(t, \varepsilon) = -x(-t, \varepsilon) = x(t + 2T, \varepsilon),\ x(t, \varepsilon) \to 0$ as $\varepsilon \to 0)$.

Because

$$x(0, \varepsilon) = x(T, \varepsilon) = 0, \tag{1.2}$$

it is clear that any solution of the boundary-value problem (1.1), (1.2), continued in t using Eq.(1.1), is odd and $2T$-periodic, i.e. it belongs to the class under consideration.

It is known that the solution of problem (1.1), (1.2) can be uniquely continued in ε if the corresponding boundary-value problem for the variational equation

$$y^{\cdot\cdot} + a(t, \varepsilon)y = 0,\ y(0) = y(T) = 0,\ a(t, \varepsilon) = f_x(x(t, \varepsilon)) \tag{1.3}$$

only has a trivial solution.

Because of conditions (1.1), $f_x(x) < 0$ for small x, and so $a(t, \varepsilon) < 0$ for small ε. In this case no solution $y(t)$ of Eq.(1.3) has more than one zero in the interval $(0, \infty)$ /4/, and so the above-specified continuability condition for $x(t, \varepsilon)$ is satisfied. Thus for small ε Eq.(1.1) has a unique periodic solution (because of the uniqueness of the extension of $x(t, \varepsilon)$) of the class under consideration. If the positional force has the monotonicity property $(f_x(x) < 0$ for all $x)$, then the solution $x(t, \varepsilon)$ exists and is unique for all ε.

If the function $f(x)$ is non-monotonic, the function $a(t, \varepsilon)$ becomes sign-varying for some value of ε. Here the continuability conditions for $x(t, \varepsilon)$ may be violated for some $\varepsilon = \varepsilon_*$, and in problem (1.3) a solution $y(t, \varepsilon_*)$ appears that is positive in the interval $(0, T)$. A further extension of $x(t, \varepsilon)$ in ε is, in general, impossible.

We shall show that if

$$p(\omega t) \geqslant 0 \text{ in } (0, T) \tag{1.4}$$

then $x(t, \varepsilon)$ increases with respect to ε in the interval $(0, T)$.

The function $x_\varepsilon(t, \varepsilon) = \partial x(t, \varepsilon)/\partial\varepsilon$ satisfies an equation obtained by differentiating (1.1) with respect to the parameter ε:

$$y^{\cdot\cdot} + a(t, \varepsilon)y + p(\omega t) = 0 \tag{1.5}$$

By virtue of (1.2)

$$x_\varepsilon(0, \varepsilon) = x_\varepsilon(T, \varepsilon) = 0 \tag{1.6}$$

The solution of the boundary-value problem (1.5), (1.6) can be expressed with the help of the corresponding Green's function $\Gamma(t, s, \varepsilon)$:

$$y(t, \varepsilon) = -\int_0^T \Gamma(t, s, \varepsilon)\, p(\omega s)\, ds \tag{1.7}$$

$$\Gamma(t, s, \varepsilon) = -\frac{y(T, s, \varepsilon)\, y(t, 0, \varepsilon)}{y(T, 0, \varepsilon)} + \delta, \quad \delta = \begin{cases} 0, & t < s \\ y(t, s, \varepsilon), & t > s \end{cases}$$

where $y(t, s, \varepsilon)$ is the solution of Eq.(1.3) satisfying the conditions $y(t, t, \varepsilon) = 0$ and $y^{\cdot}(t, t, \varepsilon) = 1$.

Because the function $y(t, 0, \varepsilon)$ has no zeros in $(0, T]$ when $\varepsilon < \varepsilon_*$, we have $\Gamma(t, s, \varepsilon) < 0$ for $t, s \in (0, T)$. Consequently $x_\varepsilon(t, \varepsilon) > 0$, i.e. $x(t, \varepsilon)$ increases with ε in the interval $(0, T)$. From this it follows in particular that under conditions (1.4) $x(t, \varepsilon) > 0$ in $(0, T)$.

Suppose $\max_t x(t, \varepsilon) = x(t_1, \varepsilon) = A(\varepsilon)$. Because at this point $x^{\cdot\cdot}(t_1, \varepsilon) \leqslant 0$, then by virtue of (1.1) $f(A(\varepsilon)) + \varepsilon p(\omega t_1) \geqslant 0$. One can show that if $\min f(x) < -\varepsilon \max p(\omega t) = -\varepsilon p_0$, then $A(\varepsilon) < A_0(\varepsilon)$, where $A_0(\varepsilon)$ is the first root of the equation $f(x) = -\varepsilon p_0$. Physically this means that the amplitude $A(\varepsilon)$ of the oscillations under consideration does not exceed the deviation $A_0(\varepsilon)$ of the system from the equilibrium position under the action of a constant force εp_0.

2. The stability of the solution under consideration is determined by the associated variational equation

$$y^{\cdot\cdot} + a(t, \varepsilon) y = 0, \quad a(t, \varepsilon) = a(t + 2T, \varepsilon) \tag{2.1}$$

As was shown above, $a(t, \varepsilon) < 0$ for small ε, and consequently Eq.(2.1) and the corresponding solution $x(t, \varepsilon)$ are unstable /4/. Of course, if $f(x)$ decreases monotonically for all x, the solution $x(t, \varepsilon)$ is unstable for all ε. Thus, in a system with a monotonically repulsive positional force, periodic oscillations cannot be physically realized.

If the function $f(x)$ is non-monotonic, then for sufficiently large ε the function $a(t, \varepsilon)$ is positive in certain intervals. As is shown below, this can lead to stability for Eq.(2.1) and, consequently, to stability for $x(t, \varepsilon)$ to a first approximation. The presence, in a realistic system, of small dissipative forces, not included in Eq.(1.1), leads to the asymptotic stability of such a solution /5/.

In applications one meets systems in which the positional force is repulsive for small x and restoring for large x (Fig.1; such properties are possessed by systems which have cracking, for example, the Mises girder). Here for $\varepsilon = 0$ we have two stable equilibrium positions ($x = c$ and $x = -c$) alongside the unstable equilibrium ($x = 0$).

We assume that the function $f(x)$ is convex for $x > 0$ and that $p(\omega t)$ satisfies condition (1.4). We will investigate changes in the stability of $x(t, \varepsilon)$ as the parameter ε increases.

Because $x(t, \varepsilon)$ increases with ε in the interval $(0, T)$, the coefficient $a(t, \varepsilon)$ also increases by virtue of the convexity of $f(x)$. At the limiting value $\varepsilon = \varepsilon_*$, the boundary-value problem (1.3) has a positive solution $y(t, \varepsilon_*)$ on $(0, T)$ as was noted above. Thus, when $\varepsilon = \varepsilon_*$ Eq.(2.1) has a $2T$-periodic solution $y(t, \varepsilon_*)$ with two zeros ($t = 0$ and $t = T$) per cycle. From the theory of the Hill equation we know /4/ that the solution at the boundaries of the second domain of instability has precisely this form. Consequently, as ε increases from zero to ε_*, system (2.1) passes in turn through the zeroth domain of instability ($0 < \varepsilon < \varepsilon_1$), the zeroth domain of stability ($\varepsilon_1 < \varepsilon < \varepsilon_2$), the first domain of instability ($\varepsilon_2 < \varepsilon < \varepsilon_3$), the first domain of stability ($\varepsilon_3 < \varepsilon < \varepsilon_4$), and, possibly, a second domain of instability (if $\varepsilon_4 < \varepsilon_*$, i.e. if ε_* corresponds to the second boundary of the indicated domain).

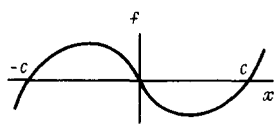

Fig.1

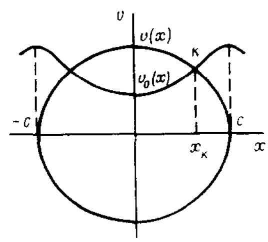

Fig.2

We shall show that the lower boundary of the zeroth domain of stability corresponds to a solution $x(t, \varepsilon)$ with amplitude $A < c$.

If the limiting amplitude $A_* = \max_t x(t, \varepsilon_*) < c$ then the assertion is obvious. Suppose that $A_* > c$, then $x(t_1, \varepsilon') = A(\varepsilon') = c$ for some $t_1 \in (0, T)$ and $\varepsilon' < \varepsilon_*$. Suppose that $x_0(t, x_0^{\cdot})$ (where $(x_0(0, x_0^{\cdot}) = 0$ and $x_0^{\cdot}(0, x_0^{\cdot}) = x_0^{\cdot})$ is a solution of the equation

$$x^{\cdot\cdot} + f(x) = 0 \tag{2.2}$$

Because $x = x^{\cdot} = 0$ is a singular point of Eq.(2.1), $x_0(t_1, x_0^{\cdot}) \to 0$ as $x_0^{\cdot} \to 0$; it is obvious that $x_0(t_1, x_0^{\cdot}) \to \infty$ as $x_0^{\cdot} \to \infty$. Hence one can find an $x_0^{\cdot}$ such that $x_0(t_1, x_0^{\cdot}) = c$. The corresponding phase trajectory $v_0(x)$ intersects the phase trajectory $v(x)$ of the solution $x(t, \varepsilon')$ at some $x = x_k$, (otherwi the equality $x_0(t_1, x_0^{\cdot}) = x(t_1, \varepsilon') = c$ would not hold).

The functions $v_0(x)$ and $v($ satisfy the equations

$$dv/dx = -v^{-1} f(x), \quad dv/dx = -v^{-1}(f(x) + P(x)), \quad P(x) = p(\omega t(x)) \tag{2.3}$$

where $t(x)$ is the inverse function to $x(t)$ $(t \in (0, t_1))$.

Since $P(x) \geqslant 0$ by virtue of (1.4), according to Chaplygin's theorem on differential inequalities $v(x) \geqslant v_0(x)$ on $(0, x_k)$ and $v(x) \leqslant v_0(x)$ in the interval (x_k, c) (Fig.2). Consequently, $x(t, \varepsilon') > x_0(t, x_0^{\cdot})$ in the interval $(0, t_1)$ and $a(t, \varepsilon) > a_0(t) = f_x(x_0(t, x_0^{\cdot}))$ because of the convexity of $f(x)$. Because of the autonomy of (2.2) $x_0^{\cdot}(t, x_0^{\cdot})$ satisfies the corresponding variational equation, i.e. for $a = a_0(t)$ Eq.(2.1) has a solution $y(t) = x_0^{\cdot}(t, x_0^{\cdot})$ with $y^{\cdot}(0) = x_0^{\cdot\cdot}(0, x_0^{\cdot}) = -f(0) = y^{\cdot}(t_1) = -f(c) = 0$ and $y(t) > 0$ in the interval $(0, t_1)$.

It can shown in exactly the same way that one can find a solution $x_1(t)$ of Eq.(2.2) such that $x_1(t_1) = c$ and $x_1(T) = 0$; the associated coefficient $a_1(t) = f_x(x_1(t)) < a(t, \varepsilon)$ in the interval (t_1, T); Eq.(2.1) for $a = a_1(t)$ has the solution $y(t) = -x_1^{\cdot}(t)$, and so $y^{\cdot}(t_1) = y^{\cdot}(T) = 0$ and $y(t) > 0$ in the interval (t_1, T).

We put $a = \Delta(t)$ in (2.1), where $\Delta(t) = a_0(t)$ in $[0, t_1]$, $\Delta(t) = a_1(t)$ in $(t_1, T]$. $\Delta(t) = \Delta(2T - t)$, $\Delta(t) = \Delta(t + 2T)$. Suppose $y(t)$ is a solution fo (2.1) for $a = \Delta(t)$ with $y^{\cdot}(t_1) = 0$ and $y(t_1) > 0$. It follows from the results obtained that $y^{\cdot}(0) = y^{\cdot}(T) = 0$, $y(t) > 0$ in the interval $[0, T]$, and $y(t) = y(2T - t) = y(t) + 2T)$, i.e. Eq.(2.1) with $a = \Delta(t)$ has a non-oscillatory $2T$-periodic solution $y(t)$.

Because $a(t, \varepsilon') > \Delta(t)$, Eq.(2.1) is oscillatory for $\varepsilon = \varepsilon'$ /4/, (i.e. the number of zeros of any solution $y(t)$ in the interval $(0, l)$ increases without limit as $l \to \infty$); consequently, the lower boundary of the zeroth domain of stability satisfies $\varepsilon_1 < \varepsilon'$. Hence, for $\varepsilon \in (\varepsilon_1, \varepsilon'')$, where $\varepsilon'' = \min(\varepsilon_2, \varepsilon')$, the solutions $x(t\ \varepsilon)$ are stable, and the corresponding amplitudes $A(\varepsilon) < c$. It follows from the latter inequality that stable periodic oscillations are also possible in a system with a positional force that is repulsive for all x, so long as the function $f(x)$ is non-monotonic.

We assume that the forcing term satisfies the condition

$$p(\omega t) = -p(\omega t + \pi) \tag{2.4}$$

as well as (1.1) and (1.4).

We remark that in applications usually $p(\omega t) = p_0 \sin \omega t$, i.e. relation (2.4) is satisfied.

Given conditions (2.4), Eq.(1.1) is satisfied by the function $-x(t + T, \varepsilon)$ as well as by the function $x(t, \varepsilon)$, and so by virtue of the uniqueness of the solution under consideration, $x(t, \varepsilon) = -x(t + T, \varepsilon)$. Using the oddness of $f(x)$ we find that $a(t, \varepsilon) = a(t + T, \varepsilon)$, i.e. the minimal period of $a(t, \varepsilon)$ is equal to T. As was remarked earlier, for $\varepsilon = \varepsilon_*$ Eq.(2.1) has a solution with period $2T$ which therefore corresponds to the boundary of the zeroth domain of instability. Hence with condition (2.4) there exists just one interval of stability $(\varepsilon_1, \varepsilon_4 \leqslant \varepsilon_*)$.

We assume in addition that

$$p^{\cdot}(\omega t) \geqslant 0 \text{ in } (0, T/2) \tag{2.5}$$

i.e. the forcing term varies monotonically between the extremal values. (In particular, this applies to $p(\omega t) = p_0 \sin \omega t$).

The same condition is also satisfied by the corresponding solution $x(t, \varepsilon)$, i.e. $x^{\cdot}(t, \varepsilon) > 0$ on $(0, T/2)$.

Indeed, differentiating (1.1) we find that the function $x^{\cdot}(t, \varepsilon) = v(t, \varepsilon)$ serves as a solution of the boundary-value problem

$$v^{\cdot\cdot} + a(t, \varepsilon) v + \varepsilon p^{\cdot}(\omega t) = 0, \quad v(-T/2, \varepsilon) = v(T/2, \varepsilon) = 0 \tag{2.6}$$

For small ε, the solution $y(t, -T/2, \varepsilon)$ (where $(y = 0$ and $y^{\cdot} > 0$ for $t = -T/2)$ of Eq.(2.1) is positive in the interval $(-T/2, T/2)$ by virtue of $a(t, \varepsilon) < 0$. As ε increases in the interval $(0, \varepsilon_*]$ the equality $y(T/2, -T/2, \varepsilon) = 0$ is impossible because here the solution $v(t, \varepsilon)$ of Eq.(2.6) would not exist (the function $p^{\cdot}(\omega t)$ and $y(t, -T/2, \varepsilon)$ are positive in the interval $(-T/2, T/2)$ and are therefore non-orthogonal). Consequently, $y(t, -T/2, \varepsilon) > 0$ and Green's function of problem (2.6), $\Gamma(t, s) < 0$ in the interval $(-T/2, T/2)$. Representing solution (2.6) in the same way as (1.7), we find that $v(t, \varepsilon) > 0$ in the interval $(-T/2, T/2)$, i.e. the coordinate $x(t, \varepsilon)$ changes monotonically between the extremal values $x(-T/2, \varepsilon) = -A(\varepsilon)$ and $x(T/2, \varepsilon) = A(\varepsilon)$.

Because $a(t, \varepsilon) = a(-t, \varepsilon)$, at the boundaries of the first domain of instability of Eq. (2.1) the periodic solutions are even or odd. The even solutions obviously satisfy the condition $y(-T/2) = y(T/2) = 0$. It was shown above that there are no such solutions when $\varepsilon = (0, \varepsilon_*]$; and so for $\varepsilon = \varepsilon_4$ Eq.(2.1) has an odd $2T$-periodic solution. Because the latter satisfies conditions (1.3), $\varepsilon_4 = \varepsilon_*$.

Thus with condition (2.5) the boundary of stability of the family $x(t, \varepsilon)$ coincides with its boundary of existence.

3. Assuming the parameter ε to be fixed, and conditions (2.4) and (2.5) to be satisfied, we will investigate the behaviour of the amplitude-frequency characteristic (AFC) $A(\omega)$ of the solution under consideration. We put $\tau = \omega t$; then Eq.(1.1) takes the form

$$\omega^2 x'' + f(x) + \varepsilon p(\tau) = 0 \tag{3.1}$$

where the prime denotes differentiation with respect to τ.

Suppose $x(\tau, A, \omega)$ is a solution of Eq.(3.1), satisfying the conditions $x = A$ and $x' = 0$ at $\tau = \pi/2$. If for some A and ω

$$x(\pi, A, \omega) = 0 \tag{3.2}$$

then the corresponding solution $x(\tau, A, \omega)$ belongs to the class under consideration. Thus relation (3.2) implicitly determines $A(\omega)$. Hence

$$dA/d\omega = -x_\omega/x_A \tag{3.3}$$

We know that the function $y(\tau) = x_A(\tau, A, \omega)$ is a solution of the corresponding variational equation

$$\omega^2 y'' + a(\tau) y = 0 \tag{3.4}$$

with $y(\pi/2) = 1$ and $y'(\pi/2) = 0$. Because $a(\tau) = a(-\tau)$, we have $y(\tau) = y(-\tau)$. It was shown above that for $\varepsilon < \varepsilon_*$ any solution of Eq.(1.3) had no more than one zero in the interval $[0, T]$; consequently $y(\tau) > 0$ in the interval $[\pi/2, \pi]$, and so $x_A > 0$.

The function $x_\omega(\tau, A, \omega)$ satisfies an equation obtained by differentiating (3.1) with respect to the parameter ω:

$$\omega^2 y'' + a(\tau) y = -2\omega x'' \tag{3.5}$$

where $x_\omega(\pi/2, A, \omega) = x_\omega'(\pi/2, A, \omega) = 0$, because $x(\pi/2, A, \omega) = A$ and $x'(\pi/2, A, \omega) = 0$. Hence, as in /5/, we find that

$$x_\omega = -\frac{2}{\omega}\int_{\pi/2}^{\pi} x''(s)\, y(\pi, s)\, ds = \frac{2}{\omega}\left[\int_{\pi/2}^{\pi} x'(s)\, y_s(\pi, s)\, ds - x'(s)\, y(\pi, s)\Big|_{\pi/2}^{\pi}\right] \tag{3.6}$$

where $y(\tau, s)$ and $y_s(\tau, s)$ are solutions of Eq.(3.4) with $y(s, s) = 0$, $y'(s, s) = 1$, $y_s(s, s) = -1$, and $y_s'(s, s) = 0$.

Because $x'(\pi/2) = 0$ and $y(\pi, \pi) = 0$, the term outside the integral in (3.6) is equal to zero. As was shown above, $x^{\cdot}(t) > 0$ in the interval $(0, T/2)$, and so $x'(\tau) < 0$ in the interval $(\pi/2, \pi)$. By virtue of the convexity of $f(x)$ we have $a(\tau) > a(z)$ for $\tau < z$ $(\tau, z \in (\pi/2, \pi))$. Because the distance between neighbouring zeros of the solution $y(\tau)$ decreases as $a(\tau)$ increases /6/, and from what was shown above, $y_s(s, \pi/2) < 0$ in the interval $[\pi/2, \pi]$, we have in addition $y_s(\pi, s) < 0$ for $s \in (\pi/2, \pi)$. Hence from (3.6) and (3.3) we find that $x_\omega > 0$ and $dA/d\omega < 0$. Thus the AFC $A(\omega)$ of the problem under consideration decreases monotonically. It can be shown that $A(\omega) \to 0$ as $\omega \to \infty$, (and so for sufficiently large ω the solution under consideration in unstable).

If $\min f(x) < -\varepsilon p_0$, then as was shown above, $A(\omega) < A_0$; (by virtue of (2.4) and (2.5) $p_0 = \max p(\omega t) = p(\pi/2)$, and so A_0 is the first root of the equation $f(x) = -\varepsilon p(\pi/2)$). In this case $a(t) < 0$, problem (1.3) only has the trivial solution, and as a result the family $x(t, \omega)$ can be continued in ω to $\omega = 0$; it is obvious that $A(\omega) \to A_0$ as $\omega \to 0$. By virtue of $a(t) < 0$ the solution $x(t, \omega)$ is unstable for all ω.

If $\min f(x) > -\varepsilon p_0$, then for small ω there are no solutions of the form being considered. Indeed, if there exists a finite limit for $A(\omega)$ as $\omega \to 0$, then $x^{\cdot\cdot}(T/2) \to 0$, $f(A) \to -\varepsilon p_0$ as $\omega \to 0$, which is, however, impossible because of the inequality $f(A) > -\varepsilon p_0$. As was shown in /7/, the period of solutions $x(t)$ of Eq.(1.1) which preserve their sign in half-periods satisfies the inequality $T(A) < T_-(A)$, where $T_-(A)$ is the period of free oscillations of the system $x^{\cdot\cdot} + f(x) - p_0 \operatorname{sgn} x = 0$ sharing with $x(t)$ the same amplitude A. Because $\lim T_-(A) = 2\pi/k$, $k^2 = \lim f(x)/x$ as $x \to \infty$ /8/, $k > 0$ by virtue of the convexity of $f(x)$, the existence of an infinite limit for $A(\omega)$ as $\omega \to 0$ is also precluded. Hence the family $x(t, \omega)$ can be continued in ω to some value $\omega_* > 0$; the corresponding problem (1.3) has a positive solution in the interval $(0, T)$. Because in this case $x_A = y(\pi, \pi/2) = 0$, we have $dA/d\omega \to -\infty$ as $\omega \to \omega_*$.

It is clear that for some frequency interval (ω_*, ω_y) the corresponding variational Eq.(2.1) is oscillatory, with $y(t, 0) > 0$ in the interval $(0, T]$. It follows from earlier considerations that for $\omega \in (\omega_*, \omega_y)$ the solution $x(t, \omega)$ is stable.

4. We will consider as an example the forced oscillations of a pendulum relative to the upper equilibrium position. The corresponding equation has the form

$$ml^2 x^{\cdot\cdot} - mgl \sin x + \varepsilon \sin \omega t = 0 \tag{4.1}$$

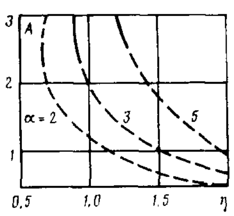

Fig.3

where x is an angular coordinate relative to the vertical, m is the mass, l is the length of the pendulum and g is the acceleration due to gravity.

For $|x| < \pi$, system (4.1) satisfies conditions (1.1), (2.4) and (2.5). From the results obtained above it follows that the solution under consideration $x(t, \varepsilon)$ satisfies the relations $x(t, \varepsilon) = -x(t + T, \varepsilon)$ and $x(t, \varepsilon) > 0$ on $(0, T)$. The AFC $A(\omega)$ decreases monotonically.

For $\varepsilon > mgl$ the solution $x(t, \varepsilon)$ exists for all ω, with $A(\omega) \to A_0 = \arcsin \varepsilon/(mgl)$ as $\omega \to 0$. For $\varepsilon > mgl$ there exists a value ω_* such that $A(\omega_*) = \pi$ or $A(\omega_*) < \pi$, $dA/d\omega \to -\infty$ as $\omega \to \omega_*$. In both cases stable solutions correspond to some frequency interval (ω_*, ω_y). Their amplitudes exceed $\pi/2$, because any solution with amplitude $A < \pi/2$ is unstable. (Here $a(t) < 0$).

Fig.3 shows the AFC $A(\eta)$, (where $(\eta = \omega/\omega_0$, $\omega_0 = (g/l)^{1/2})$, obtained by numerical methods for various values of the parameter $\alpha = \varepsilon/(mgl)$. The solid lines represent stable solutions and the dashed lines unstable ones. It can be seen that as α increases the length of the interval (ω_*, ω_y), corresponding to stable solutions with amplitude $A < \pi$, increases. The amplitude behaviour of the solutions under consideration, which depends on the dimensionless frequency η and the parameter α characterizing the magnitude of the forcing term, completely agrees with the theoretical results established above.

As can be seen from Fig.3, for sufficiently small α $(\alpha < \alpha_* \simeq 3)$ there is a frequency interval in which there exists a second solution $x_2(t, \varepsilon)$ of the type under consideration with amplitude $A_2 < \pi$, coinciding with $x(t, \varepsilon)$ for $\omega = \omega_*$. The solution $x_2(t, \varepsilon)$ is unstable; unlike $x(t, \varepsilon)$ its amplitude increases with ω and decreases with ε.

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THE PERTURBED MOTIONS OF A SOLID CLOSE TO REGULAR LAGRANGIAN PRECESSIONS*

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The asymptotic behaviour of a Lagrange gyroscope under the influence of a weak perturbing moment is investigated for the case of motions that are close to regular precessions. An averaged system of equations of motion is obtained in special evolutionary variables. The cases of a small constant moment and the presence of a cavity filled with highly viscous fluid are considered in detail.

1. The equations of motion and statement of the problem. The motion of a heavy axisymmetric rigid body with a fixed point on the axis of symmetry (a Lagrange gyroscope) under the influence of a perturbing mechanical moment of arbitrary nature is described by the equations

$$\Omega_1^{\cdot} = -(\lambda\Omega_3 - \Omega_2 \operatorname{ctg}\vartheta)\Omega_2 + \varkappa \sin\vartheta + \varepsilon M_1 \quad (1.1)$$
$$\Omega_2^{\cdot} = (\lambda\Omega_3 - \Omega_2 \operatorname{ctg}\vartheta)\Omega_1 + \varepsilon M_2$$

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